MATH1010E University Mathematics
Quiz 1
Suggested Solutions

1. (a)

$$\begin{aligned}\frac{d}{dx}\left(\frac{x\sin 2x}{1+x^2}\right) &= \frac{(1+x^2)(\sin 2x + 2x\cos 2x) - (x\sin 2x)(2x)}{(1+x^2)^2} \\ &= \frac{(1-x^2)\sin 2x + 2x(1+x^2)\cos 2x}{(1+x^2)^2}.\end{aligned}$$

(b)

$$\begin{aligned}\frac{d}{dx}\exp(\sqrt{1+\sin^2 x}) &= \exp(\sqrt{1+\sin^2 x})\,\frac{1}{2}\frac{2\sin x\cos x}{\sqrt{1+\sin^2 x}} \\ &= \frac{\sin x\cos x}{\sqrt{1+\sin^2 x}}\,\exp(\sqrt{1+\sin^2 x}).\end{aligned}$$

(c)

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 4x}{\sqrt{4+x}-\sqrt{4-x}} &= \lim_{x\to 0}\frac{\sin 4x(\sqrt{4+x}+\sqrt{4-x}}{(4+x)-(4-x)} \\ &= \lim_{x\to 0}\frac{\sin 4x(\sqrt{4+x}+\sqrt{4-x}}{2x} \\ &= \lim_{x\to 0}\frac{\sin 4x}{4x}\cdot\lim_{x\to 0}2(\sqrt{4+x}+\sqrt{4-x}) \\ &= 1\cdot 2(2+2) \\ &= 8\end{aligned}$$

(d) For any $x \neq 0$, we have the following inequality

$$-\sqrt{|x|} \leq \sqrt{|x|}\cos\left(\frac{x\ln|x|}{x-\sin x}\right) \leq \sqrt{|x|}.$$

Since we have

$$\lim_{x\to 0} -\sqrt{|x|} = 0 = \lim_{x\to 0}\sqrt{|x|},$$

by sandwich theorem, we have

$$\lim_{x\to 0}\sqrt{|x|}\cos\left(\frac{x\ln|x|}{x-\sin x}\right) = 0.$$

2. From the definition of derivative as limit of difference quotient,

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{\frac{1}{\sqrt{x+h+1}}-\frac{1}{\sqrt{x+1}}}{h} \\
&= \lim_{h\to 0}\frac{\sqrt{x+1}-\sqrt{x+h+1}}{h\sqrt{x+1}\sqrt{x+h+1}} \\
&= \lim_{h\to 0}\frac{(x+1)-(x+h+1)}{h\sqrt{x+1}\sqrt{x+h+1}(\sqrt{x+1}+\sqrt{x+h+1})} \\
&= \lim_{h\to 0}\frac{-h}{h\sqrt{x+1}\sqrt{x+h+1}(\sqrt{x+1}+\sqrt{x+h+1})} \\
&= \lim_{h\to 0}\frac{-1}{\sqrt{x+1}\sqrt{x+h+1}(\sqrt{x+1}+\sqrt{x+h+1})} \\
&= \frac{-1}{\sqrt{x+1}\sqrt{x+1}(\sqrt{x+1}+\sqrt{x+1})} \\
&= -\frac{1}{2(x+1)^{3/2}}.
\end{aligned}$$

3. (a) For $x>0$, $f(x)=x^2\sin(\ln x)$, hence

$$f'(x)=2x\sin(\ln x)+x^2\cos(\ln x)\cdot\frac{1}{x}=2x\sin(\ln x)+x\cos(\ln x).$$

For $x<0$, $f(x)=x^2\sin(\ln -x)$, hence

$$f'(x)=2x\sin(\ln(-x))+x^2\cos(\ln(-x))\cdot\frac{1}{x}=2x\sin(\ln(-x))+x\cos(\ln(-x)).$$

Therefore, for $x\neq 0$,

$$\begin{aligned}
f'(x) &= 2x\sin(\ln|x|)+x\cos(\ln|x|) \\
&= \begin{cases} 2x\sin(\ln x)+x\cos(\ln x) & \text{when } x>0, \\ 2x\sin(\ln(-x))+x\cos(\ln(-x)) & \text{when } x<0. \end{cases}
\end{aligned}$$

(b) Using the definition of derivative,

$$\begin{aligned}
f'(0) &= \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} \\
&= \lim_{x\to 0}\frac{x^2\sin(\ln|x|)-0}{x-0} \\
&= \lim_{x\to 0}x\sin(\ln|x|) \\
&= 0.
\end{aligned}$$

The last equality holds by sandwich theorem since $-|x|\le x\sin(\ln|x|)\le|x|$ for all $x\neq 0$ and that $\lim_{x\to 0}-|x|=\lim_{x\to 0}|x|=0$.

(c) From (a), $f'(x) = 2x\sin(\ln|x|) + x\cos(\ln|x|)$. Moreover, by sandwich theorem, we have

$$\lim_{x\to 0} 2x\sin(\ln|x|) = 0 = \lim_{x\to 0} x\cos(\ln|x|).$$

Therefore,

$$\lim_{x\to 0} f'(x) = 0.$$

On the other hand, $f'(0) = 0$ by (b). Hence, $\lim_{x\to 0} f'(x)$ exists and equals to $f'(0)$, which means that $f'(x)$ is continuous at $x = 0$.

(d) Consider the difference quotient,

$$\frac{f'(x) - f'(0)}{x - 0} = \frac{2x\sin(\ln|x|) + x\cos(\ln|x|)}{x} = 2\sin(\ln|x|) + \cos(\ln|x|).$$

Since the limit of the right hand side does not exist for $x \to 0$ (this can be seen by taking $x = e^{-(2n+1)\pi/2}$ for $n = 0, 1, 2, 3, \ldots$), this implies that f' is not differentiable at $x = 0$.

— End of Solutions to Quiz 1 —